

# Distribution of points of interpolation and of zeros of exact maximally convergent multipoint Padé approximants

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**Abstract:** Given a regular compact set  $E$  in  $\mathbb{C}$ , a unit measure  $\mu$  supported by  $\partial E$ , a triangular point set  $\beta := \{\{\beta_{n,k}\}_{k=1}^n\}_{n=1}^\infty$ ,  $\beta \subset \partial E$  and a function  $f$ , holomorphic on  $E$ , let  $\pi_{n,m}^{\beta,f}$  be the associated multipoint  $\beta$ -Padé approximant of order  $(n, m)$ . We show that if the sequence  $\pi_{n,m}^{\beta,f}$ ,  $n \in \Lambda$ ,  $m$ -fixed, converges exact maximally to  $f$ , as  $n \rightarrow \infty$ ,  $n \in \Lambda$  inside the maximal domain of  $m$ -meromorphic continuability of  $f$  relatively to the measure  $\mu$ , then the points  $\beta_{n,k}$  are uniformly distributed on  $\partial E$  with respect to the measure  $\mu$  as  $n \in \Lambda$ . Furthermore, a result about the zeros behavior of the exact maximally convergent sequence  $\Lambda$  is provided, under the condition that  $\Lambda$  is "dense enough."

**Keywords:** Multipoint Padé approximants, maximal convergence

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## 1 Introduction

We first introduce some needed notations.

Let  $\Pi_n$ ,  $n \in \mathbb{N}$  be the class of the polynomials of degree  $\leq n$  and  $\mathcal{R}_{n,m} := \{r = p/q, p \in \Pi_n, q \in \Pi_m, q \not\equiv 0\}$ .

Given a compact set  $E$ , we say that  $E$  is *regular*, if the unbounded component of the complement  $E^c := \overline{\mathbb{C}} \setminus E$  is solvable with respect to Dirichlet problem. We will assume throughout the paper that  $E$  possesses a connected complement  $E^c$ . In what follows, we will be working with the max-norm  $\|\dots\|_E$  on  $E$ ; that is  $\|\dots\|_E := \max_{z \in E} |\dots|(z)$ .

Let  $\mathcal{B}(E)$  be the class of the unit measures supported on  $E$ ; that is  $\text{supp}(\dots) \subseteq E$ . We say that the infinite sequence of Borel measures  $\{\mu_n\} \in \mathcal{B}(E)$  converges in the weak topology to a measure  $\mu$  and write  $\mu_n \longrightarrow \mu$ , if

$$\int g(t) d\mu_n \rightarrow \int g(t) d\mu$$

for every function  $g$  continuous on  $E$ . We associate with a measure  $\mu \in \mathcal{B}(E)$  the logarithmic potential  $U^\mu(z)$ ; that is,

$$U^\mu(z) := \int \log \frac{1}{|z - t|} d\mu.$$

Recall that  $U^\mu$  ([13]) is a function superharmonic in  $\mathbb{C}$ , subharmonic in  $\overline{\mathbb{C}} \setminus \text{supp}(\mu)$ , harmonic in  $\mathbb{C} \setminus \text{supp}(\mu)$  and

$$U^\mu(z) = \ln \frac{1}{|z|} + o(1), z \rightarrow \infty.$$

We also note the following basic fact ([2]):

**Carleson's lemma:** *Given the measures  $\mu_1, \mu_2$  supported by  $\partial E$ , suppose that  $U^{\mu_1}(z) = U^{\mu_2}(z)$  for every  $z \notin E$ . Then  $\mu_1 = \mu_2$ .*

Finally, we associate with a polynomial  $p \in \Pi_n$  the normalized counting measure  $\mu_p$  of  $p$ , that is

$$\mu_p(F) := \frac{\text{number of zeros of } p \text{ on } F}{\deg p},$$

where  $F$  is a point set in  $\mathbb{C}$ .

Given a domain  $B \subset \mathbb{C}$ , a function  $g$  and a number  $m \in \mathbb{N}$ , we say that  $g$  is  $m$ -meromorphic in  $B$  ( $g \in \mathcal{M}_m(B)$ ) if it has no more than  $m$  poles in  $B$  (poles are counted with their multiplicities). We say that a function  $f$  is holomorphic on the compactum  $E$  and write  $f \in \mathcal{A}(E)$ , if it is holomorphic in some open neighborhood of  $E$ .

Let  $\beta$  be an infinite triangular table of points,  $\beta := \{\{\beta_{n,k}\}_{k=1}^n\}_{n=1,2,\dots}$ ,  $\beta_{n,k} \in E$ , with no limit points outside  $E$ . Set

$$\omega_n(z) := \prod_{k=1}^n (z - \beta_{n,k}).$$

Let  $f \in \mathcal{A}(E)$  and  $(n, m)$  be a fixed pair of nonnegative integers. The rational function  $\pi_{n,m}^{\beta,f} := p/q$ , where the polynomials  $p \in \Pi_n$  and  $q \in \Pi_m$  are such that

$$\frac{fq - p}{\omega_{n+m+1}} \in \mathcal{A}(E)$$

is called a  $\beta$ -multipoint Padé approximant of  $f$  of order  $(n, m)$ . As is well known, the function  $\pi_{n,m}^{\beta,f}$  always exists and is unique ([14],[3]). In the particular case when  $\beta \equiv 0$ , the multipoint Padé approximant  $\pi_{n,m}^{\beta,f}$  coincides with the classical Padé approximant  $\pi_{n,m}^f$  of order  $(n, m)$  ([12]).

Set

$$\pi_{n,m}^{\beta,f} := \frac{P_{n,m}^{\beta,f}}{Q_{n,m}^{\beta,f}}, \quad (1)$$

where the polynomials  $P_{n,m}^{\beta,f}$  and  $Q_{n,m}^{\beta,f}$  do not have common divisors. The zeros of  $Q_{n,m}^{\beta,f}$  are called *free zeros* of  $\pi_{n,m}^{\beta,f}$ ;  $\deg Q_{n,m} \leq m$ .

We say that the points  $\beta_{n,k}$  are *uniformly distributed relatively to the measure  $\mu$* , if

$$\mu_{\omega_n} \longrightarrow \mu, \quad n \rightarrow \infty.$$

We recall the notion of  $m_1$ -Hausdorff measure (cf. [4]). For  $\Omega \subset \mathbb{C}$ , we set

$$m_1(\Omega) := \inf \left\{ \sum_{\nu} |V_{\nu}| \right\}$$

where the infimum is taken over all coverings  $\{V_{\nu}\}$  of  $\Omega$  by disks and  $|V_{\nu}|$  is the radius of the disk  $V_{\nu}$ .

Let  $D$  be a domain in  $\mathbb{C}$  and  $\varphi$  a function defined in  $D$  with values in  $\overline{\mathbb{C}}$ . A sequence of functions  $\{\varphi_n\}$ , meromorphic in  $D$ , is said to converge to a function  $\varphi$   *$m_1$ -almost uniformly inside  $D$*  if for any compact subset  $K \subset D$  and every  $\varepsilon > 0$  there exists a set  $K_{\varepsilon} \subset K$  such that  $m_1(K \setminus K_{\varepsilon}) < \varepsilon$  and the sequence  $\{\varphi_n\}$  converges uniformly to  $\varphi$  on  $K_{\varepsilon}$ .

For  $\mu \in \mathcal{B}(E)$ , define

$$\rho_{\min} := \inf_{z \in E} e^{-U^{\mu}(z)}$$

and

$$\varrho_{\max} := \max_{z \in E} e^{-U^{\mu}(z)};$$

( $U^{\mu}$  is superharmonic on  $E$ ; hence it attains its minimum (on  $E$ )). As is known ([15], [13]),

$$e^{-U^{\mu}(z)} \geq \rho_{\min}, \quad z \in E^c.$$

Set, for  $r > \rho_{\min}$ ,

$$E_{\mu}(r) := \{z \in \mathbb{C}, e^{-U^{\mu}(z)} < r\}.$$

Because of the upper semicontinuity of the function  $\chi(z) := e^{-U^{\mu}(z)}$ , the set  $E_{\mu}(r)$  is open; clearly  $E_{\mu}(r_1) \subset E_{\mu}(r_2)$  if  $r_1 \leq r_2$  and  $E_{\mu}(r) \supset E$  if  $r > \varrho_{\max}$ .

Let  $f \in \mathcal{A}(E)$  and  $m \in \mathbb{N}$  be fixed. Let  $R_{m,\mu}(f) = R_{m,\mu}$  and  $D_{m,\mu}(f) = D_{m,\mu} := E_\mu(R_{m,\mu})$  denote, respectively, the radius and domain of  $m$ -meromorphy with respect to  $\mu$ ; that is

$$R_{m,\mu} := \sup\{r, f \in \mathcal{M}_m(E_\mu(r))\}.$$

Furthermore, we introduce the notion of a  $\mu$ -maximal convergence to  $f$  with respect to the  $m$ -meromorphy of a sequence of rational functions  $\{r_{n,\nu}\}$  (a  $\mu$ -maximal convergence): that is, for any  $\varepsilon > 0$  and each compact set  $K \subset D_m$ , there exists a set  $K_\varepsilon \subset K$  such that  $m_1(K \setminus K_\varepsilon) < \varepsilon$  and

$$\limsup_{n+\nu \rightarrow \infty} \|f - r_{n,\nu}\|_{K_\varepsilon}^{1/n} \leq \frac{\|e^{-U^\mu}\|_K}{R_{m,\mu}(f)}.$$

Hernandez and Calle Ysern proved the following:

**Theorem A, [6]** : Let  $E, \mu, \beta$  and  $\omega_n, n = 1, 2, \dots$ , be defined as above. Suppose that  $\mu_{\omega_n} \rightarrow \mu$  as  $n \rightarrow \infty$  and  $f \in \mathcal{A}(E)$ . Then, for each fixed  $m \in \mathbb{N}$ , the sequence  $\pi_{n,m}^{\beta,f}$  converges to  $f$   $\mu$ -maximally with respect to the  $m$ -meromorphy.

Theorem A generalizes E. B. Saff's theorem of Montessus de Ballore's type about multipoint Padé approximants (see[14]).

We now utilize the normalization of the polynomials  $Q_{n,m}(z)$  with respect to a given open set  $D_{m,\mu}$ ; that is,

$$Q_{n,m}(z) = \prod (z - \alpha'_{n,k}) \prod (1 - z/\alpha''_{n,k}), \quad (2)$$

where  $\alpha'_{n,k}, \alpha''_{n,k}$  are the zeros lying inside, resp. outside  $D_{m,\mu}$ . Under this normalization, for every compact set  $K$  and  $n$  large enough there holds

$$\|Q_{n,m}^{\beta,f}\|_K \leq C_1,$$

where  $C_1 = C_1(K)$  is a positive constant, depending on  $K$ . In the sequel, we denote by  $C_i$  positive constant, independent on  $n$  and different at different occurrences.

In [6], the set  $K_\varepsilon$  (look at the definition of a  $\mu$ -maximal convergence) is explicitly written, namely  $K_\varepsilon := K \setminus \Omega(\varepsilon)$ , where

$$\Omega(\varepsilon) := \bigcup_{n=1}^{\infty} \left( \bigcup_{\alpha'_{n,k}} \{z, |z - \alpha'_{n,k}| < \varepsilon/(2mn^2)\} \right).$$

For  $\Omega(\varepsilon)$  we have

$$m_1(\Omega(\varepsilon)) \leq \varepsilon.$$

For points  $z \notin \Omega(\varepsilon)$ , we have

$$|Q_{n,m}^{\beta,f}(z)| \geq C_2(\varepsilon/mn^2)^{k_n},$$

where  $k_n$  stands for the number of the zeros of  $Q_{n,m}^{\beta,f}$  in  $D_{m,\mu}$ ;  $k_n \leq m$ .

Let  $Q$  be the monic polynomial, the zeros of which coincide with the poles of  $f$  in  $D_{m,\mu}$ ;  $\deg Q \leq m$ . It was proved in [6] (Proof of Lemma 2.3) that for every compact subset  $K$  of  $D_{m,\mu}$

$$\limsup_{n \rightarrow \infty} \|fQQ_{n,m}^{\beta,f} - QP_{n,m}^{\beta,f}\|_K^{1/n} \leq \frac{\|e^{-U^\mu}\|_K}{R_{m,\mu}}. \quad (3)$$

Hence, the function  $-U^\mu(z) - \ln R_{m,\mu}$  is a harmonic majorant in  $D_{m,\mu}$  of the family  $\{|(fQQ_{n,m}^{\beta,f} - QP_{n,m}^{\beta,f})(z)|^{1/n}\}_{n=1}^\infty$ .

**Theorem B, [6]** *With  $E, \mu, m, \omega_n$  and  $f$  as in Theorem A, assume that  $K$  is a regular compact set for which  $\|e^{-U^\mu}\|_K$  is not attained at a point on  $E$ . Suppose that the function  $f$  is defined on  $K$  and satisfies*

$$\limsup_{n \rightarrow \infty} \|f - \pi_{n,m}^{\beta,f}\|_K^{1/n} \leq \|e^{-U^\mu}\|_K / R < 1.$$

Then  $R \leq R_{m,\mu}(f)$ .

**Remark:** Suppose that  $\infty > R_{m,\mu} > \varrho_{\max}$  and  $D_{m,\mu}$  is connected. Let  $V$  be a disk in  $D_m \setminus E_\mu(\varrho_{\max})$ , centered at a point  $z_0$  of radius  $r > 0$  and such that  $f$  is analytic on  $V$ . Fix  $r_1$ ,  $0 < r_1 < r$  and set  $A := \{z, r_1 \leq |z - z_0| \leq r\}$ . Fix a number  $\varepsilon < (r - r_1)/4$ . Introduce, as before, the set  $\Omega(\varepsilon)$ . Recall that

$$m_1(\Omega(\varepsilon)) \leq \varepsilon.$$

It is clear that the set  $A \setminus \Omega(\varepsilon)$  contains a concentric circle  $\Gamma$  (otherwise we would obtain a contradiction with  $m_1(\Omega(\varepsilon)) < (r - r_1)/4$ .) We note that the function  $f$  and the rational functions  $\pi_{n,m}^{\beta,f}$  are well defined on  $\Gamma$ . Viewing (3), we may write

$$\limsup_{n \rightarrow \infty} \|QQ_{n,m}^{\beta,f}f - QP_{n,m}^{\beta,f}\|_\Gamma^{1/n} \leq \|e^{-U^\mu}\|_\Gamma / R_{m,\mu}.$$

Suppose that

$$\limsup_{n \rightarrow \infty} \|QQ_{n,m}^{\beta,f}f - QP_{n,m}^{\beta,f}\|_\Gamma^{1/n} < \|e^{-U^\mu}\|_\Gamma / R_{m,\mu},$$

or, what is the same,

$$\limsup_{n \rightarrow \infty} \|QQ_{n,m}f - QP_{n,m}^{\beta,f}\|_\Gamma^{1/n} \leq \|e^{-U^\mu}\|_\Gamma / (R_{m,\mu} + \sigma) < 1$$

for an appropriate  $\sigma > 0$ . Then,

$$|(f - \pi_{n,m}^{\beta,f})(z)|_\Gamma \leq C_3(n^2m/\varepsilon)^m(\|e^{-U^\mu}\|_\Gamma / (R_{m,\mu} + \sigma))^n$$

for all  $z \in \Gamma$  and  $n$  large enough. This leads to

$$\limsup_{n \rightarrow \infty} \|f - \pi_{n,m}^{\beta,f}\|_{\Gamma}^{1/n} \leq \|e^{-U^\mu}\|_{\Gamma} / (R_{m,\mu} + \sigma).$$

using Theorem B, we arrive at  $R_{m,\mu} + \sigma < R_{m,\mu}$ . The contradiction yields

$$\limsup_{n \rightarrow \infty} \|QQ_{n,m}^{\beta,f}f - QP_{n,m}^{\beta,f}\|_{\overline{V}_\Gamma}^{1/n} = \|e^{-U^\mu}\|_{\overline{V}_\Gamma} / R_{m,\mu},$$

where  $V_\Gamma$  is the disk bounded by  $\Gamma$ .

Then the function  $-U^\mu - \ln R_{m,\mu}$  is an exact harmonic majorant of the family  $\{|fQQ_{n,m}^{\beta,f} - QP_{n,m}^{\beta,f}|^{1/n}\}$  in  $D_{m,\mu}$  (see (3)). Therefore, there exists a subsequence  $\Lambda$  such that for every compact subset  $K \subset D_{m,\mu} \setminus E$

$$\lim_{n \in \Lambda} \|QfQ_{n,m}^{\beta,f} - P_{n,m}^{\beta,f}Q\|_K^{1/n} = \|e^{-U^\mu}\|_K / R_{m,\mu}. \quad (4)$$

(see [16],[17]) for a discussion of exact harmonic majorant)). We will refer to this sequences as to *an exact maximally convergent sequence*.

We prove

**Theorem 1:** *Under the same conditions on  $E$ , assume that  $\mu \in \mathcal{B}(\partial E)$  and that  $\beta \subset \partial E$  is a triangular set of points. Let  $m \in \mathbb{N}$  be fixed,  $f \in \mathcal{A}(E)$  and  $\varrho_{\max} < R_{m,\mu} < \infty$ . Suppose that  $D_{m,\mu}$  is connected. If for a subsequence  $\Lambda$  of the multipoint Padé approximants  $\pi_{n,m}^{\beta,f}$  condition (4) holds, then  $\mu_{\omega_n} \rightarrow \mu$  as  $n \rightarrow \infty$ ,  $n \in \Lambda$ .*

The problem of the distribution of the points of interpolation of multipoint Padé approximants was investigated, so far, only for the case when the measure  $\mu$  coincides with the equilibrium measure  $\mu_E$  of the compact set  $E$ . It was first raised by J. L. Walsh ([18], Chp. 3) while considering maximally convergent polynomials with respect to the equilibrium measure. He showed that the sequence  $\mu_{\omega_n}$  converges weakly to  $\mu_E$  through the entire set  $\mathbb{N}$  (respectively their associated measures onto the boundary of  $E$ ) iff the interpolating polynomials of every function  $f_t(z)$  of the form  $f_t(z) := (t - z), t \notin E, t -$  fixed, converge maximally to  $f_t$ . Walsh's result was extended to multipoint Padé approximants with a fixed number of the free poles by N. Ikononov in [8], as well as to generalized Padé generalized approximants, associated with a regular condenser ([7]). The case of polynomial interpolation of an arbitrary function  $f$  holomorphic in  $E$  was considered by R. Grothmann ([5]); he established the existence of an appropriate sequence  $\Lambda$  such that  $\mu_{\omega_n} \rightarrow \mu_E$ ,  $n \rightarrow \infty$ ,  $n \in \Lambda$ , respectively the balayage measures onto  $\partial E$ . Grothmann's result was generalized in relation to multipoint Padé approximants  $\pi_{n,m}^{\beta,f}$  with a fixed number of the free poles (see [9]). Finally, in [1] was considered the case when the degrees of the denominators tend slowly to infinity, namely  $m_n = o(n/\ln n)$ .

As a consequence of Theorem 1, we derive

**Theorem 2:** *Under the conditions of Theorem 1, suppose that the exact maximally convergent sequence  $\Lambda := \{n_k\}_{k=1}^\infty$  satisfies the condition to be "dense enough"; that is*

$$\limsup \frac{n_{k+1}}{n_k} < \infty.$$

*Then there is at least one point  $z_0 \in \partial D_{m,\mu}(f)$  such that for every disk  $V_{z_0}(r)$  centered at  $z_0$  of radius  $r$*

$$\limsup_{n \rightarrow \infty, n \in \Lambda} \mu_{P_{n,m}^{\beta,f}}(V_{z_0}(r)) > 0, \text{ as } n \rightarrow \infty, n \in \Lambda.$$

**Proof of Theorem 1:** Set  $Q_{n,m}^{\beta,f} := Q_n$ ,  $P_{n,m}^{\beta,f} := P_n$  and  $F := fQ$ . Fix numbers  $R, \tau, r$  such that  $\varrho_{\max} < R < \tau < r < R_{m,\mu}$  and  $E_\mu(R)$  is connected. Then, by the conditions of the theorem, for every compactum  $K \subset D_{m,\mu}$  (comp.(4))

$$\lim_{n \in \Lambda} \|FQ_n - QP_n\|_K^{1/n} = \|e^{-U^\mu}\|_K / R_{m,\mu}, n \in \Lambda. \quad (5)$$

Select a positive number  $\eta$  such that  $R + \eta < \tau < \tau + \eta < r < R_{m,\mu}$ . Let  $\Gamma$  be an analytic curve in  $E_\mu(r) \setminus E_\mu(\tau + \eta)$  such that  $\Gamma$  winds around every point in  $E_\mu(\tau)$  exactly once. In an analogous way, we select a curve  $\gamma \subset E_\mu(R + \eta) \setminus E_\mu(R)$ . Additionally, we require that  $U^\mu$  is constant on  $\Gamma$  and  $\gamma$ . Set

$$F_n(z) := \frac{1}{n} \ln |FQ_n - P_nQ|(z) + U^\mu(z) + \ln R_{m,\mu}, n \in \Lambda. \quad (6)$$

Let  $\sigma > 0$  be arbitrary. The functions  $F_n$  are subharmonic in  $E_\mu(r) \setminus E_\mu(R)$ . By (5) and the choice of  $\Gamma$ ,

$$\max_{t \in \Gamma} F_n(t) \leq -\min_{t \in \Gamma} + \max_{t \in \Gamma} + \sigma \leq \sigma, N \in \Lambda, n \geq n_1, n_1(\sigma)$$

and, analogously,

$$\max_{t \in \gamma} F_n(t) \leq -\min_{t \in \gamma} + \max_{t \in \gamma} \leq \sigma, N \in \Lambda, n > n_1.$$

Then, by the max-principle of subharmonic functions,

$$\max_{z \in \mathcal{A}_{\gamma,\Gamma}} F_n(z) \leq \sigma, n \in \Lambda, n \geq n_1, N \in \Lambda, \quad (7)$$

where  $\mathcal{A}_{\gamma,\Gamma}$  is the "annulus", bounded by  $\Gamma$  and  $\gamma$ .

On the other hand, by (5), for any compact set  $K \subset E_r \setminus E_R$  and  $n$  large enough there is a point  $z_{n,K} \in K$  such that

$$-\min_K U^\mu(z_{n,K}) - \ln R_{m,\mu} - \sigma \leq \frac{1}{n} \ln |FQ_n(z_{n,K}) - QP_n(z_{n,K})|, n \geq n_3(K), n \in \Lambda.$$

Therefore,

$$-\sigma \leq F_n(z_{n,K}), n \geq n_2(K, \sigma). \quad (8)$$

Further, by the formula of Hermite-Lagrange, for  $z \in \gamma$  we have

$$FQ_n(z) - QP_n(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega_{n+m+1}(z)}{\omega_{n+m+1}(t)} \frac{FQ_n(t) - QP_n(t)}{t - z} dt.$$

Hence, by (5),

$$\begin{aligned} & \frac{1}{n} \ln |FQ_n(z) - QP_n(z)| \leq \\ & \max_{t \in \Gamma} U^{\omega_{n+m+1}}(t) - U^{\omega_{n+m+1}}(z) + \frac{1}{n} \ln \|FQ_n - QP_n\|_{\Gamma} + \frac{1}{n} \text{const} \leq \\ & \max_{t \in \Gamma} U^{\omega_{n+m+1}}(t) - U^{\omega_{n+m+1}}(z) - \min_{t \in \Gamma} U^{\mu}(t) - \ln R_{m,\mu} + \sigma, n \in \Lambda, n \geq n_3 = n_3(\sigma) > n_1, \end{aligned}$$

where  $U^{\omega_{n+m+1}} := U^{\mu_{\omega_{n+m+1}}}$ . To simplify the notations, we set  $U^{\omega_{n+m+1}} := U^{\omega_n}$ . ( The correctness will be not lost, since  $m \in \mathbb{N}$  is fixed). Involving into consideration the functions  $F_n$  (see (6)), we get for  $z \in \gamma$

$$\begin{aligned} F_n(z) & \leq \max_{t \in \Gamma} (U^{\omega_n}(t) - U^{\mu}(t)) + \max_{t \in \Gamma} U^{\mu}(t) + \\ & (U^{\mu}(z) - U^{\omega_n}(z)) - \min U^{\mu}(t) + \sigma, n \in \Lambda, n \geq n_2 \geq n_1. \end{aligned}$$

By Helly's selection theorem ([13]), there exists a subsequence of  $\Lambda$  which we denote again by  $\Lambda$  such that  $\mu_{\omega_{n+m+1}} := \mu_{\omega_n} \longrightarrow \omega, n \in \Lambda$ . Passing to the limit, we obtain

$$\limsup_{\Lambda} |F_n(z)| \leq \max_{t \in \Gamma} (U^{\omega}(t) - U^{\mu}(t)) + (U^{\mu}(z) - U^{\omega}(z)), z \in \gamma. \quad (9)$$

Consider the function  $\phi$ , harmonic in  $\mathcal{A}_{\Gamma, \gamma}$  and

$$\phi := \begin{cases} 0, & \Gamma, \\ \min(0, -\min_{t \in \gamma} (U^{\mu}(t) - U^{\omega}(t)) + (U^{\mu}(z) - U^{\omega}(z))), & \gamma \end{cases}$$

From (7) and (9), we arrive at

$$\limsup F_n(z) \leq \phi,$$

for  $z$  in  $\mathcal{A}_{\Gamma, \gamma}$ . Being harmonic,  $\phi$  obeys the maximum and the minimum principles in this region. The definition yields

$$\phi(z) \leq 0, z \in \mathcal{A}_{\Gamma, \gamma}$$

We will show that

$$\phi(z) \equiv 0, \quad (10)$$

Suppose that (10) is not true. Let  $\Upsilon$  be a closed curve in the set  $E_{R+\eta} - \gamma^o$ , where  $\gamma^o$  stands for the interior of  $\gamma$ . Then there exists a number  $\theta > 0$  such



that  $\phi \leq -\Theta$  for every  $z \in \Upsilon$ . This inequality contradicts (8), for  $\sigma$  close enough to the zero and  $n \in \Lambda$  sufficiently large. .

Hence,  $\phi \equiv 0$ . Then the definition of  $\phi$  yields

$$U^\mu(z) - U^\omega(z) \equiv \min_{t \in \gamma} (U^\mu(t) - U^\omega(t)), \quad z \in \gamma.$$

The function  $U^\mu(z) - U^\omega(z)$  is harmonic in the unbounded complement  $G$  of  $\gamma$ , and by the maximum principle,

$$U^\mu(z) - U^\omega(z) \equiv \text{Constant}, \quad z \in G;$$

consequently,

$$U^\mu(z) - U^\omega(z) \equiv \text{Constant}, \quad z \in E^c.$$

On the other hand,  $(U^\mu - U^\omega)(\infty) = 0$ , which yields  $U^\mu \equiv U^\omega$  in  $E^c$ . By Carleson's Lemma,  $\mu = \omega$ . On this, Theorem 1 is proved. **Q.E.D.**

The proof of Theorem 2 will be preceded by an auxiliary lemma

**Lemma 1, [10] :** *Given a domain  $U$ , a regular compact subset  $S$  and a sequence  $\vartheta := \{n_k\}$  of positive integers,  $n_k < n_{k+1}$ ,  $k = 1, 2, \dots$ , such that*

$$\limsup \frac{n_{k+1}}{n_k} < \infty.$$

*Suppose that  $\{\phi_{n_k}\}$  is a sequence of rational functions,  $\phi_{n_k} \in \mathcal{R}_{n_k, n_k}$ ,  $k = 1, 2, \dots$ ,  $\phi_{n_k} = \phi'_{n_k} / \phi''_{n_k}$  having no more than  $m$  poles in  $U$  and converging uniformly of  $\partial S$  to a function  $\phi \not\equiv 0$  such that*

$$\limsup_{n_k \rightarrow \infty, n_k \in \Lambda} \|\phi_{n_k} - \phi\|_{\partial S}^{1/n_k} < 1.$$

*Assume, in addition, that on each compact subset of  $U$*

$$\lim \mu_{\phi'_{n_k}}(K) \longrightarrow 0. \tag{11}$$

*Then the function  $\phi$  admits a continuation into  $U$  as a meromorphic function with no more than  $m$  poles.*

**Proof of Theorem 2.** We preserve the notations in the proof of Theorem 1.

The proof of Theorem 2 follows from Lemma 1 and Theorem 1. Indeed, under the conditions of the theorem the sequence  $\{\pi_n\}_{n \in \Lambda}$  converges maximally to  $f$  with respect to the measure  $\mu$  and the domain  $D_{m, \mu}$ . Hence, inside  $D_{m, \mu}$  condition (11) is fulfilled. From the proof of Theorem 1, we see that there is a regular compact subset  $S$  of  $D_{m, \mu}$  such that  $\limsup_{n \in \Lambda} \|f - \pi_n\|_S^{1/n} < 1$ .

Suppose now that the statement of Theorem 2 is not true. Then there is an open strip  $W$  containing  $\partial D_{m, \mu}$  such that on each compact subset of  $W$

condition (11) holds. Applying Lemma 1 with respect to the sequence  $\pi_n$  and the domain  $D_{m,\mu} \cup W$ , we conclude that  $f \in \mathcal{M}_m(\overline{D_{m,\mu}})$ . This contradicts the definition of  $D_{m,\mu}$ .

On this, the proof of Theorem 2 is completed.

**Q.E.D.**

Using again Lemma 1 and applying Theorem A, we obtain a result related to the zero distribution of the sequence  $\{\pi_{n,m}^{\beta,f}\}$ .

**Theorem 3:** *Let  $E$  be a regular compactum in  $\mathbb{C}$  with a connected complement, let  $\mu \in \mathcal{B}(E)$  and  $\beta \in E$  be a triangular point set. Let the polynomials  $\omega_n, n = 1, 2, \dots$ , be defined as above. Suppose that  $\mu_{\omega_n} \rightarrow \mu$  as  $n \rightarrow \infty$  and  $f \in \mathcal{A}(E)$ . Let  $m \in \mathbb{N}$  be fixed, and suppose that  $R_{m,\mu} < \infty$ . Then, there is at least one point  $z_0 \in \partial D_{m,\mu}$  such that  $\limsup_{n \rightarrow \infty} \mu_{\pi_{n,m}^{\beta,f}}(\overline{V}_{z_0}(r)) > 0$  for every positive  $r$ .*

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